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Nonconservation of the Net Current of Dirac Particles in Tolman–Bondi and Robertson–Walker Geometries

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A straightforward calculation shows that, in contrast to what happens for the Dirac equation in the Kerr metric, the net current of particles is not conserved in the case of the Dirac equation in the Tolman–Bondi and Robertson–Walker space-times.

1. INTRODUCTION

It is well known that the formulation of the Dirac equation can be extended to curved space-time by using the notions of covariant derivatives and of Pauli generalized σ-matrices. In particular this can be done by means of the spinorial formalism that has been developed after the pioneering paper by Newman and Penrose (1962), an account of which can be found in the books by Penrose and Rindler (1986) and Chandrasekhar (1983). In the following we adopt Chandrasekhar's notations and mathematical conventions. According to this formulation, the Dirac equation reads

$$\sigma^{\alpha}_{AA'} P^{A}_{;\alpha} + i\mu_{*} \overline{Q}_{A'} = 0$$

$$\sigma^{\alpha}_{AA'} Q^{A}_{;\alpha} + i\mu_{*} \overline{P}_{A'} = 0 \qquad (1)$$

$\mu_{*}\sqrt{2}$ is the mass of the particle and P^{A} and $\overline{Q}^{A'}$ are spinors representing the wave function. If l, n, m, m^{*} is the null tetrad frame of the Newman–Penrose formalism in a given metric, the generalized σ-matrices are defined by

$$\sigma^{\alpha}_{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} l^{\alpha} & m^{\alpha} \\ m^{*\alpha} & n^{\alpha} \end{pmatrix} \qquad (2)$$

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Associated to the Dirac equation there is the spinorial current

$$J^{AA'} = P^A\overline{P}^{A'} + Q^A\overline{Q}^{A'} \tag{3}$$

The connection between the components in the local coordinate basis and the spinorial components of the current is expressed by

$$J^\alpha = \sqrt{2}\sigma^\alpha_{AA'}J^{AA'} \tag{4}$$

The current conservation

$$J^\alpha_{;\alpha} = 0 \tag{5}$$

is a direct consequence of the Dirac equation (1) and the relations (4), (3).

In connection with the physical interpretation, it is worth noting that the positivity of J^0 is ensured by the positivity of the matrix $\sigma^0_{AA'}$. This is provided in general by construction from the tetrad frame and it holds true in our cases of interest, namely for the Robertson–Walker and Tolman–Bondi metrics. In the case of the Kerr metric this property can be checked to hold directly from the representation of $\sigma^0_{AA'}$ given in Chandrasekhar (1983).

Of course, from the four-dimensional Gauss law, one can give an integral form to the conservation law (5).

However, we are interested here in three-dimensional considerations.

By a standard result, equation (5) is equivalent to (see, e.g., Schutz, 1990)

$$\partial_t(\sqrt{-g}J^t) = -\partial_k(\sqrt{-g}J^k) \tag{6}$$

As a consequence we have the three-dimensional integral relation

$$\partial_t\left(\int_V dx_1\, dx_2\, dx_3\, \sqrt{-g}J^t\right) = -\oint dS_2\, \sqrt{-g}J^h u_h \tag{7}$$

S is the surface surrounding the spatial region V.

In the case of one-particle theory, a Schrödinger-like statistical interpretation would then follow from (7) by choosing the solution of equation (1) such that J^h vanishes at infinity. If, however, the theory is such that particle creation is possible, the physical solutions can no longer be chosen to make the right-hand side of equation (7) vanish, so that a Schrödinger-like interpretation is not possible.

This is the case for the Dirac equation in the Robertson–Walker metric [see Parker (1971); for further developments see Birrell and Davies (1982)] and in the Kerr metric (Starobinskii, 1973; Unruh, 1974; Wald, 1976).

Also in the Tolman–Bondi model there is in principle particle creation, this effect being in general a property of the Dirac equation in time-dependent gravitational fields (see, e.g., Birrell and Davies, 1982). This is also a consequence of the fact that the Tolman–Bondi model contains solutions which describe collapsing universes leading to black hole formation (Demianski and Lasota, 1973) and it is known that particle creation near black holes occurs (Hawking, 1975; Wald, 1975).

One is therefore interested in the net current of particles, a quantity we now denote by $\partial N/\partial t$. With a suitable choice of V in equation (7) one has

$$\frac{\partial N}{\partial t} = -\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sqrt{-g} J^r \tag{8}$$

J^r is the radial component of the Dirac current. It is a fact that such a quantity is conserved in the case of the Dirac equation in Kerr metric. A central role is played to that end by the time independence of the metric coefficients and by the complete separability of the Dirac equation (Chandrasekhar, 1983).

It is the object of this paper to show that the net current of particles given in equation (8) is not conserved in the Robertson–Walker or in the Tolman–Bondi geometry. This seems to be a consequence of the fact that, in both cases, the Dirac equation is not separable in its r and t dependences. The result could be of interest in connection with the problem of particle formation in the early universe of the standard cosmology (Kolb and Turner, 1990).

2. TOLMAN–BONDI GEOMETRY

The metric of the Tolman–Bondi geometry is given by

$$ds^2 = dt^2 - e^{\Gamma} dr^2 - Y^2(d\theta^2 + \sin^2\theta \, d\phi^2) \tag{9}$$

where $\Gamma = \Gamma(r, t)$, $Y = Y(r, t) > 0$.

For the purposes of the following considerations the explicit expressions of the functions Γ, Y are not necessary. It is sufficient to consider them as functions fixed by the underlying cosmological model. For instance, they could be given by the solution of the Tolman–Bondi model (Tolman, 1934; Bondi, 1947), an account of which in the Newman–Penrose formalism can be found in Zecca (1993a). This model consists of a spherically symmetric space-time filled with dust matter of zero pressure described, in a comoving coordinate system, by the metric (9).

We assume the null tetrad frame to be given by

$$l^i = \frac{1}{\sqrt{2}}(1, e^{-\Gamma/2}, 0, 0)$$

$$n^i = \frac{1}{\sqrt{2}}(1, -e^{-\Gamma/2}, 0, 0)$$

$$m^i = \frac{1}{\sqrt{2}Y}(0, 0, 1, i \csc \theta)$$

$$m^{*i} = \frac{1}{\sqrt{2}Y}(0, 0, 1, -i \csc \theta) \tag{10}$$

In terms of the directional derivatives $D = l^i\partial_i$, $\Delta = n^i\partial_i$, $\delta = m^i\partial_i$, and $\delta^* = m^{*i}\partial_i$ and of the nonzero Ricci rotation coefficients, the Dirac equations assume the form (Chandrasekhar, 1983)

$$\begin{aligned}
(D + \epsilon - \rho)F_1 + (\delta^* - \alpha)F_2 &= i\mu_* G_1 \\
(\Delta + \mu - \gamma)F_2 + (\delta - \alpha)F_1 &= i\mu_* G_2 \\
(D + \epsilon - \rho)G_2 - (\delta - \alpha)G_1 &= i\mu_* F_2 \\
(\Delta + \mu - \gamma)G_1 - (\delta^* - \alpha)G_2 &= i\mu_* F_1
\end{aligned} \tag{11}$$

where we set $P^A \equiv (F_1, F_2)$, $\overline{Q}^{A'} \equiv (-G_2, G_1)$. The nonzero spin coefficients corresponding to the tetrad frame (10) have values

$$\begin{aligned}
\rho &= -\frac{1}{\sqrt{2}Y}(\dot{Y} + Y'e^{-\Gamma/2}) \\
\mu &= \frac{1}{\sqrt{2}Y}(\dot{Y} - Y'e^{-\Gamma/2}) \\
\beta &= -\alpha = \frac{\cot \theta}{2\sqrt{2}Y} \\
\epsilon &= -\gamma = \frac{\dot{\Gamma}}{4\sqrt{2}}
\end{aligned} \tag{12}$$

(The overdot and prime denote here partial derivatives with respect to t and r.) As usual, owing to the symmetry of the metric, the ϕ dependence can be chosen to be of the form $e^{im\phi}$ ($m = 0, \pm 1, \pm 2, \pm 3, \ldots$). With this assumption, equations (11) become

$$\begin{aligned}
\sqrt{2}Y(D + \epsilon - \rho)F_1 + L^- F_2 &= i\mu_* Y G_1 \sqrt{2} \\
\sqrt{2}Y(\Delta + \mu - \gamma)F_2 + L^+ F_1 &= i\mu_* Y G_2 \sqrt{2} \\
\sqrt{2}Y(D + \epsilon - \rho)G_2 - L^+ G_1 &= i\mu_* Y F_2 \sqrt{2} \\
\sqrt{2}Y(\Delta + \mu - \gamma)G_1 - L^- G_2 &= i\mu_* Y F_1 \sqrt{2}
\end{aligned} \tag{13}$$

where

$$L^{\pm} = \partial_\theta \mp m \csc \theta + \frac{1}{2}\cot \theta$$

and the F's and G's are now functions of r, θ, t. The θ dependence can be separated by using the Chandrasekhar–Teukolski method (see, e.g., Chandrasekhar, 1983). With

$$F_1 = f_1(r, t)S_1(\theta), \qquad F_2 = f_2(r, t)S_2(\theta)$$
$$G_1 = f_2(r, t)S_1(\theta), \qquad G_2 = f_1(r, t)S_2(\theta) \tag{14}$$

inserted in equation (13), one gets for S_1, S_2 the angular equations

$$L^-S_2 = -\lambda S_1, \qquad L^+S_1 = \lambda S_2 \tag{15}$$

λ is a separation constant. By setting

$$H_1 = Yf_1, \qquad H_2 = Yf_2$$

after the separation, the functions H_1, H_2 are found to satisfy

$$DH_1 + \epsilon H_1 = \left(i\mu_* + \frac{\lambda}{Y\sqrt{2}}\right)H_2$$
$$\Delta H_2 + \epsilon H_2 = \left(i\mu_* - \frac{\lambda}{Y\sqrt{2}}\right)H_1 \tag{16}$$

A detailed solution of equation (15) can be found in Montaldi and Zecca (1994). For the present purposes it is sufficient to recall that λ is real and that if $|m| \geq 1$, then $\lambda^2 = (l + 1/2)^2$, $l = 1, 2, 3, \ldots$, with S_1, S_2 being essentially the Jacobi polynomials, while if $m = 0$, then $\lambda^2 = (l + 1)^2$, $l = 0, 1, 2, 3, \ldots$, and S_1, S_2 are essentially the Tchebychef polynomials of the second kind.

In general equations (16) cannot be separated in the r and t dependences. This in particular holds in the Tolman–Bondi model, where the explicit solution Y itself (see, e.g., Demianski and Lasota, 1973) cannot be written as a function of r times a function of t.

In order to perform the calculation of the expression (8), we have preliminarily

$$\sigma^r_{AA'} = \frac{e^{-\Gamma/2}}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{17}$$

$$J^r = \frac{e^{-\Gamma/2}}{2}(|S_1|^1 + |S_2|^2)\frac{(|H_1|^2 - |H_2|^2)}{Y^2} \tag{18}$$

Then we obtain

$$\frac{\partial N}{\partial t} = 2\pi(|H_2|^2 - |H_1|^2) \tag{19}$$

where the S functions have been normalized to one. To see that the term on the right-hand side of equation (19) is not constant, consider the expressions

$$A = |H_2|^2 - |H_1|^2, \qquad B = |H_2|^2 + |H_1|^2 \tag{20}$$

By making equations (16) explicit with respect to ∂_t, ∂_r, one gets

$$\dot{A} = \frac{2\sqrt{2}}{Y}\left[H_1\overline{H}_2\left(i\mu_* Y - \frac{\lambda}{\sqrt{2}}\right) + c.c.\right] - 2\sqrt{2}\epsilon A + e^{-\Gamma/2}B' \tag{21}$$

and

$$A' = e^{\Gamma/2}(2\sqrt{2}\epsilon B + \dot{B}) \tag{22}$$

Since the component of the spinor of the Dirac equation (1) satisfies a Klein–Gordon-like equation, it has a continuous dependence together with its derivatives on the initial data (Wald, 1984). Suppose now $\dot{A} = A' = 0$; then from Eq. (22), B comes out to be a function determined by the given ϵ structure. But this is impossible as a consequence of equation (21) and the arbitrariness of the initial data of H_1, H_2 in the Dirac equation.

3. THE ROBERTSON–WALKER GEOMETRY

The metric is given in this case by

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1 - ar^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \tag{23}$$

Also here the function $R(t)$ is understood to be a fixed function given by the underlying cosmological model, which could be, for instance, the standard cosmology of the Friedman–Einstein model (see, e.g., Weinberg, 1972).

Here we choose as null tetrad frame

$$\begin{aligned}
l^i &= \frac{1}{\sqrt{2}}\left(1, \frac{(1 - ar^2)^{1/2}}{R}, 0, 0\right) \\
n^i &= \frac{1}{\sqrt{2}}\left(1, -\frac{(1 - ar^2)^{1/2}}{R}, 0, 0\right) \\
m^i &= \frac{1}{\sqrt{2}rR}(0, 0, 1, i\csc\theta) \\
m^{*i} &= \frac{1}{\sqrt{2}rR}(0, 0, 1, -i\csc\theta) = \overline{m^i}
\end{aligned} \tag{24}$$

The corresponding nonzero spin coefficients were obtained in Montaldi and Zecca (1994) to be

$$\rho = -\frac{1}{\sqrt{2}}\left[\frac{\dot{R}}{R} + \frac{(1 - ar^2)^{1/2}}{rR}\right]$$

$$\mu = \frac{1}{\sqrt{2}}\left[\frac{\dot{R}}{R} - \frac{(1 - ar^2)^{1/2}}{rR}\right]$$

$$\beta = -\alpha = \frac{\cot\theta}{2\sqrt{2}rR}$$

$$\epsilon = -\gamma = \frac{\dot{R}}{2\sqrt{2}R} \tag{25}$$

With these values and by mimicking the method of the previous section, we can separate part of the wave function of the Dirac equation (11) by the position (14), obtaining in this way exactly the same results (15).

For the r, t dependences we are left with the analog of equation (16):

$$DH_1 + \epsilon H_1 = \left(i\mu_* - \frac{\lambda}{rR\sqrt{2}}\right)H_2$$

$$\Delta H_2 + \epsilon H_2 = \left(i\mu_* + \frac{\lambda}{rR\sqrt{2}}\right)H_1 \tag{26}$$

where now $H_1 = rRf_1$, $H_2 = rRf_2$.

By using the explicit expression of ϵ and of the directional derivatives $D = 2^{-1/2}[\partial_t + (1 - ar^2)^{1/2}/(rR)\partial_r]$ and $\Delta = 2^{-1/2}[\partial_t - (1 - ar^2)^{1/2}/(rR)\partial_r]$, it is not difficult to show that the solutions H_1, H_2 of equations (26) cannot be separated in the r and t dependences.

With the notations of the previous section, we have

$$\sigma^r_{AA'} = \frac{(1 - ar^2)^{1/2}}{2R}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{27}$$

$$J^r = \frac{(1 - ar^2)^{1/2}}{2R}\left(|S_1|^2 + |S_2|^2\right)\frac{|H_1|^2 - |H_2|^2}{r^2R^2} \tag{28}$$

so that also here we have

$$\frac{\partial N}{\partial t} = 2\pi(|H_2|^2 - |H_1|^2) = 2\pi A(r, t) \tag{29}$$

The equations corresponding to (21), (22) are essentially the same:

$$\dot{A} = \frac{2\sqrt{2}}{rR}\left[H_2\overline{H}_1\left(\frac{\lambda}{\sqrt{2}} - i\mu_* rR\right) + c.c.\right] - 2\sqrt{2}\epsilon A + \frac{(1 - ar^2)^{1/2}}{R} B' \tag{30}$$

$$A' = \frac{R}{(1 - ar^2)^{1/2}}(2\sqrt{2}\epsilon B + \dot{B}) = \frac{\partial}{\partial t}\left(\frac{BR}{(1 - ar^2)^{1/2}}\right) \tag{31}$$

and the conditions $\dot{A} = A' = 0$ are not possible, by the same argument as in the previous section. This can be seen here also in an elementary direct way. Indeed, equation (31) can be separated [in contrast to the case of equation (22)] by

$$A = f(r)T(t), \qquad B = g(r)S(t) \tag{32}$$

to get the relation

$$g = kf'(1 - ar^2)^{1/2} \tag{33}$$

k is the separation constant. If now $A' = 0$, then by equation (33), $B = 0$ and hence $H_1 = H_2 = 0$.

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